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Very Large Gaps between Consecutive Primes*

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Let $G(X)$ denote the largest gap between consecutive primes below X. Improving earlier results of Erdős, Rankin, Schönhage, and Maier-Pomerance, we prove

where $\log_\nu X$ denotes the ν-fold iterated logarithm function and γ is Euler's constant. The new tool used is a combinatorial result proved by probabilistic methods. © 1997 Academic Press

1

Let p, p' denote consecutive primes and let

$$G(X) := \max_{p \leqslant X} (p' - p) \qquad \text{for} \quad X \geqslant 2 \tag{1.1}$$

denote the largest gap between consecutive primes below X.

Cramér [1] conjectured using probabilistic arguments

$$\limsup_{X \to \infty} \frac{G(X)}{\log^2 X} = 1. \tag{1.2}$$

Some recent results indicate that the above limsup might be bigger (perhaps infinity) since the irregularities in the distribution of primes are more significant than expected by Cramér's model. Nevertheless these arguments would still suggest $G(X) = 0(\log^{2+\varepsilon} X)$; that is, we believe that (1.2) is not too far from the truth.

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The prime number theorem implies the bound $G(X) \geqslant (1+o(1)) \log X$ and Erdős [2] obtained

$$G(X) \gg \frac{\log X \log_2 X}{\log_3^2 X} \tag{1.3}$$

in 1935, where $\log_\nu X$ denotes the ν-fold iterated logarithm function. This was improved three years later by Rankin [5] to (γ is Euler's constant)

$$G(X) \geqslant c_0(e^\gamma + 0(1)) \frac{\log X \log_2 X \log_4 X}{\log_3^2 X}, \tag{1.4}$$

where his first value for c_0 was $c_0 = e^{-\gamma}/3$. It took further 25 years to reach the value $c_0 = 1$ (Rankin [6]) in 1963, which was in some sense the natural bound of the Erdős–Rankin method (see Section 2).

Erdős [3] offered \$10,000 for a proof of (1.4) with arbitrarily large c at the meeting in Durham in 1979. Nevertheless, there was no improvement on the value of c_0 until 1990, when Maier and Pomerance [4] could show that (1.4) holds with

$$c_0 = 1.31256..., \tag{1.5}$$

breaking the natural bound $c_0 = 1$ of Rankin. The proof uses besides the earlier tools a very deep analytic number theory, a sort of Bombieri–Vinogradov-type theorem for the generalized twin prime problem. An essential further point of the proof is a combinatorial theorem proved by Maier and Pomerance.

As in all of the mentioned results, the lower bounds were proved through the Jacobstahl function $j(n)$, defined as the maximal gap between consecutive integers coprime to n. If

$$J(X) = \max_{n \leqslant X} j(n), \tag{1.6}$$

then it is simple to see that for $X \geqslant 7$

$$G(X) \geqslant J(X). \tag{1.7}$$

Our aim in this work is to show how the arguments of Maier and Pomerance can be modified to show the same result with $c_0 = 2$. That is, we will prove

THEOREM 1. $G(X) \geqslant J(X) \geqslant 2(e^\gamma + o(1))(\log X \log_2 X \log_4 X / \log_3^2 X)$.

The improvement refers exclusively to the combinatorial part of their work, Theorem 4.1′. Substituting this in [4] by our Theorem 2 (cf.

Section 3) we immediately get our Theorem 1. We will outline this in the next section.

2

Although the crucial Theorem 2 (cf. Section 3) is of purely combinatoric nature (as Theorem 4.1′ in [4]), the benefit of its use in the arithmetic problem for $J(X)$ can be explained quite clearly. By using Theorem 2 the construction of a long interval $I=[s+1, ..., s+U]$ with the property

$$\left(s+i, \prod_{p \leqslant t} p\right) > 1 \qquad \text{for} \quad i = 1, 2, ..., U \tag{2.1}$$

is much more effective; that is, we can assure an interval more than 50% longer than in [4]. Further, in contrast to Theorem 4.1′, our Theorem 2 is essentially optimal and in that way we reach a new natural bound of the Erdős–Rankin–Maier–Pomerance method with $c_0 = 2$. Further on we sketch this construction (see Section 2 of [4] for more details).

Let $j'(n)$ denote the largest integer u such that there exists a residue system $a_p \pmod p$ for all $p \mid n$ with the property that every integer $i \in [1, u]$ satisfies at least one of the congruences $i \equiv a_p \pmod p$. Then $j'(n) = j(n) - 1$ from the Chinese Remainder Theorem (cf. [4]). So (changing the variable $\log X$ to x) our task reduces to show that

$$j'(P(x) \geqslant (c_0 e^{\gamma} + o(1))\, x \log x \log_3 x (\log_2 x)^{-2} \qquad \text{with} \quad c_0 = 2 \tag{2.2}$$

where $P(x) = \prod_{p \leqslant x} p = e^{(1+o(1))\,x}$.

This is shown by choosing

$$\begin{aligned} a_p &= 1 \qquad \text{for every prime} \quad p \leqslant y = \exp\left\{\frac{(1-\varepsilon)\log x \log_3 x}{\log_2 x}\right\}, \\ a_p &= 0 \qquad \text{for every prime} \quad p \in (y, z] \qquad \text{where} \quad z = \frac{x}{\log_2 x}, \end{aligned} \tag{2.2}$$

and a_p "optimally" for every prime $p \in (z, x]$.

Let $U = c'x \log x \log_3 x (\log_2^{-2} x)$ with an arbitrary $c' < c_0$. Then, as shown in Section 5 of [4], after sieving out all elements $i \in [1, U]$ which satisfy at least one congruence $i \equiv a_p \pmod p$ with $p \leqslant z$, the residual set $R^*(R \cup R'$in [4]) has cardinality

$$|R^*| \sim \frac{c'}{1-\varepsilon} \frac{x}{\log x}. \tag{2.4}$$

Now, if we use all the primes $p \in (z, x]$ in the trivial way to sieve out at least one element in R^*, we can show (2.2) with $c_0 = 1$, since the number of primes $p \in (z, x]$ is clearly

$$\pi(x) - \pi(z) \sim \frac{x}{\log x} \geqslant |R^*| \sim \frac{c'}{1-\varepsilon} \frac{x}{\log x} \tag{2.5}$$

if $c' = (1-\varepsilon)^2$, for example. This is, briefly speaking, Rankin's argument. To illustrate the difficulties in improving Rankin's bound $c_0 = 1$, we mention (cf. Section 2 of [4]) that at the present state of number theory it seems to be impossible to show for any fixed prime $q \in (z, x)$ that there are two members of R^* that are congruent mod q. This explains why we called $c_0 = 1$ a natural bound of Rankin's method.

The breaking through of Maier and Pomerance [4] was that they showed

(i) there are many (in fact, the expected number) congruent pairs in R^* mod q for almost all primes $q \in Q = \{q \in (z, x]; q \text{ prime}\}$,

(ii) we can simultaneously choose distinct pairs $r_1(q)$, $r_2(q) \in R^*$, $r_1(q) \equiv r_2(q) \pmod q$ for at least 31% of all primes $q \in Q$ (that is we have a subset $\tilde{Q} \subset Q$, $|\tilde{Q}|/|Q| > 0.31$, such that for $q', q'' \in \tilde{Q}$, $r_i(q') \neq r_j(q'')$ for all combinations of $i = 1, 2$ and $j = 1, 2$).

The assertion (i) was proved in [4] via the analytic number theoretical methods used in Goldbach's problem and in the problem of the generalized twin primes. The second assertion (ii) used, besides all the mentioned arithmetical information, the relatively simple combinatorial Theorem 4.1′ (cf. Section 4 of [4]).

In order to prove our Theorem 1 we do not need more arithmetic information. The use of Theorem 2 (cf. Section 3) instead of Theorem 4.1′ of [4] enables us to show that

(ii′) we can simultaneously choose distinct pairs $r_1(q)$, $r_2(q) \in R^*$, $r_1(q) \equiv r_2(q) \pmod q$ for almost all primes $q \in Q$, that is, (ii) holds with a subset $Q' \subset Q$ with $|Q'| > (1-\varepsilon)\,|Q|$.

Since we do not see any possibility of showing that for at least one $q_0 \in Q$ we would have $r_1(q), r_2(q), r_3(q) \in R^*$ with $r_1(q) \equiv r_2(q) \equiv r_3(q) \pmod q$ (although we believe this to be true) it seems to be very difficult to reach any constant $c_0 > 2$ in (1.4) by applying the Erdős–Rankin–Maier–Pomerance method. In such a way we can say that the value $c_0 = 2$ gives a new natural bound for this method.

It is of methodological interest that the purely combinatorial Theorem 2 (which is needed for the proof of the arithmetical Theorem 1) will be shown by probabilistic methods.

3

In order to formulate our combinatorial result we begin with the

Definition. A graph $\mathcal{G}$ is N-colored if there is a function χ from the edge set G of $\mathcal{G}$ to $\{1, ..., N\}$.

We think of $1, ..., N$ as different colors and $\chi(e)$ as the color of the edge e.

In the application the role of the different colors will be played by the primes $q \in Q$ (i.e., $q \in (z, x]$) and the vertices of the graph will be elements of R^* (or more precisely, of R). Two elements of type mp and mp' from R^* will be connected with an edge of color q, if $|p - p'| = k_0 q$ where k_0 is a fixed number during the procedure. According to this there are at most two edges of the same color at any given vertex A of $\mathcal{G}$. But, as one can show that it happens quite rarely that $p - k_0 q$, p, $p + k_0 q$ (and q) are all primes, often deleting these edges all the other conditions remain valid and we may suppose that there is at most one edge of each color at any given vertex.

In order to make our combinatorial argument more simple and clear we prefer to restrict the proof for the case when we have only at most one edge of each color at any vertex. We suppose that the cardinality of the vertex set V of $\mathcal{G}$ is

$$|V| = M = cN \qquad 2 \leqslant c \leqslant c^* \tag{3.1}$$

with a constant c^*. We will use the following definitions:

Definition. A set $\mathcal{E}$ of edges is called pairwise independent, or briefly independent, if each vertex of the graph is incident to at most one edge of $\mathcal{E}$.

Definition. A set $\mathcal{E}$ of independent edges is called a partial matching.

Definition. A totally multicolored subgraph (TMC) of a graph is defined as a subgraph where all the edges have distinct colors. We are looking for a big TMC partial matching, possibly with $(1 - \varepsilon) N$ edges.

In order to define the conditions for our graph we will introduce a further definition.

Definition. Let $\delta \geqslant 0$ be arbitrary. An N-colored graph $\mathcal{G}$ with $M = cN$ vertices $(2 \leqslant c \leqslant c^*)$ is called (S, F, δ)-regular if

(i) for every vertex A there is at most one edge of each color incident to A,

(ii) for all but δN colors each color v in $\{1, ..., N\}$ is assigned to a set $G(v)$ of edges with

$$S(v) = |G(v)| \in ((1-\delta)\, S, (1+\delta)\, S), \tag{3.2}$$

(iii) for all but δM vertices, the number $m(A)$ of edges incident to a vertex A, the degree of the vertex, satisfies

$$m(A) \in \left((1-\delta)\frac{2S}{c}, (1+\delta)\frac{2S}{c} \right). \tag{3.3}$$

(iv) for the exceptional vertices $m(A) \leqslant F \cdot 2S/c$.

THEOREM 2. *For any* $c^* \geqslant 2$, $F > 0$, *and* $\eta > 0$ *there exist* $\delta = \delta(c^*, F, \eta) > 0$ *and* $B(\eta, F, \delta)$ *such that each* (S, F, δ)*-regular* N *colored graph having* $M = cN$ *vertices with* $c \in [2, c^*]$ *has a partial matching* $\mathscr{E}$ *with distinct colors* (*a TMC partial matching*) *satisfying the inequality*

$$|\mathscr{E}| > (1-\eta)\, N \tag{3.4}$$

if $\min(S, N) > B(\eta, F, c^*)$.

Choosing

$$K = \left[\frac{4}{\eta}\right]^2, \tag{3.5}$$

we are able to substitute (iii) with a seemingly stronger condition

(iii′) the N colors can be partitioned into K classes $L^1, ..., L^K$ in such a way that each class contains $(N/K)(1+\vartheta)$ colors with $|\vartheta| \leqslant \delta$, and for all but $2\,\delta M$ vertices the number $m(A, t)$ of edges of L_t incident to a vertex A satisfies

$$m(A, t) \in \left((1-2\delta)\frac{2S}{cK}, (1+2\delta)\frac{2S}{cK} \right) \tag{3.6}$$

for all values $t = 1, ..., K$.

This was actually the form of conditions for how the original Theorem 4.1′ of [4] was formulated, and the condition (iii′) was true in the case of the given application. The author is indebted for the suggestion by Professor I. Z. Ruzsa that instead of (iii′) it is enough to suppose the simpler condition (iii) because (iii) implies (iii′). In order to see this it is sufficient to partition every color mutually independently with probability $1/K$ into each color class L^t with $t = 1, 2, ..., K$. Then by the law of large numbers, (iii′) will be true with probability $1 - o(1)$ as $\min(S, N) \to \infty$, where the $o(1)$ function might depend on δ, c_1, and K. Nevertheless, the formulation of Theorem 2 is definitely simpler if we do not need to introduce the partition of colors into classes.

In what follows we will slightly reformulate our final Theorem 2. Let $c^* \geqslant 2$, $F \geqslant 1$ be constants, K be a natural number, $\min(N, S) > B'(c^*, F, K)$. The symbol $o(1)$ replaces an arbitrary function tending to 0 as $\min(S, N) \to \infty$. (This function may depend on c^*, F and K.) Assume that the $N(1 + o(1))$ colors are divided into K classes L^t ($t = 1, 2, ..., K$), each one containing $(N/K)(1 + o(1))$ colors, and suppose the graph has $(1 + o(1))\, M$ vertices, where $M = cN$, $2 \leqslant c \leqslant c^*$. We remark that although we will use the properties (i″), (ii″), (iii″), and (iv″) a number of times for different subgraphs of the original graph $\mathcal{G}$ with different values of c_ν, F_ν, K_ν, N_ν, S_ν, the number of steps will be bounded (less than K), and $c_\nu \leqslant Kc^*$, $F_\nu \leqslant K \cdot F$, $\min(N_\nu, S_\nu) \geqslant (1/K)\, \min(N, S)$ so we may use the same symbol $o(1)$ during the whole procedure. The expressions almost all vertices, colors, etc., refer to exceptional sets of size $o(M)$, $o(N)$, etc.

Thus we will actually prove the following form of Theorem 2.

Theorem 2′. *Assume a graph $\mathcal{G}$ has the following properties*:

(i″) *for every vertex A and color v there is at most one edge incident to A with color v*,

(ii″) *almost all colors v are assigned to a set $G(v)$ of edges with*

$$S(v) = |G(v)| = (1 + o(1))\, S, \tag{3.3}$$

(iii″) *for almost all vertices A the number of edges incident to A and colored with colors belonging to L^t satisfy*

$$m(A, t) = (1 + o(1))\, \frac{2S}{cK} \qquad \textit{for all} \quad t = 1, 2, ..., K, \tag{3.4}$$

(iv″) *for the possible $o(M)$ exceptional vertices A we have for the degree of A*

$$m(A) \leqslant F \cdot \frac{2S}{c},$$

with the restrictions above for the parameters c, F, K, S, M, N.

Then there exists a partial matching $\mathscr{E}$ with distinct colors (*a TMC partial matching*) *of size* $|\mathscr{E}| > (1 - 4/\sqrt{K})\, N$.

We remark that in the application for our arithmetic problem, M and N are of order $x(m \log x)^{-1}$ where $m \in (1, U/z)$, and S is of order $U(m \log^2 x)^{-1} \log_3 x$ whilst the $o(1)$ function could be explicitly given as $(\log_3 x)^{-1}$; that is, in the definition we could take $\delta = (\log_3 x)^{-1}$. These values follow from Section 4 of [4].

Proof. Let us denote for any color v the vertex set of the edges in $G(v)$ by $V(v)$, and let $G^t = \bigcup_{v \in L^t} G(v)$.

A color will be called regular if it satisfies (3.3), and irregular otherwise. (ii′) implies that $\sum_{1 \leqslant v \leqslant L,\, v \text{ regular}} S(v) = (1 + o(1))\, NS$.

On the other hand, we have from (iii″) and (iv″)

$$\sum_{1 \leqslant v \leqslant L} S(v) = \frac{1}{2} \sum_{A \in V} m(A) = \frac{1}{2} (1 + o(1))\, M \cdot \frac{2S}{c} = (1 + o(1))\, NS.$$

So we obtain $\sum_{1 \leqslant v \leqslant L,\, v \text{ irregular}} S(v) = o(NS)$.

This relation shows that irregular colors have practically no effect on (iii″). More precisely, if $m^*(A, t)$ denotes the number of edges incident to A and colored with any regular color in class L^t then we have similarly to (iii″) for almost all vertices A,

$$m^*(A, t) = (1 + o(1)) \frac{2S}{cK} \qquad \text{for all} \quad t = 1, 2, ..., K. \tag{3.5}$$

We will call a vertex regular if it satisfies (3.5), and irregular otherwise. An edge will be called regular if both of its vertices are regular. We will call a color v good if it is regular and almost all edges in $G(v)$ are regular. Since the $o(M)$ irregular points have at most $O(S)$ neighbours each, the total number of irregular edges is $o(MS) = o(NS)$, therefore almost all regular colors (and a fortiori almost all colors) are good. Finally, we will call a vertex A good if A is regular and almost all of its neighbours are regular; more precisely, if there are only $o(S)$ irregular points B such that $(A, B) \in G$. Since the total number of irregular edges is $o(MS)$, almost all vertices $A \in V$ are good.

We will use $E(\xi)$ and $D^2(\xi)$ for expectation and variance resp. of the random variable ξ.

4

Our strategy for the proof will be the following. First we choose independently from every regular color v in the first color-class L^1 one edge randomly where all the edges of color v have the same probability $1/S(v)$. Although we cannot expect all $1+o(1))(N/K)$ edges to be independent, the expected number of edges with higher multiplicity will be only of order N/K^2 which is just a negligible $O(1/K)$ portion of all edges. (This would not be true naturally if we chose simultaneously one edge of each color randomly.) After this set H_1 of vertices (including the multiple vertices too) is chosen, we delete all edges in the first color-class, as well as all edges of other colors having a vertex in H_1. In such a way we obtain a subgraph with about $M_1 = Me^{-2/K_c}$ vertices, colored with $N_1 = N(1-1/K)$ colors in $K_1 = K-1$ color classes and we will show that it has the same properties (i″)–(iv″) for other parameters M_1, N_1, K_1, S_1, c_1. This is trivially true for (i″) and with a slightly increased value F_1 for (iv″), so our task is reduced to showing that the important regularity properties (ii″)–(iii″) will be preserved with probability $(1-o(1))$ for the remaining graph $\mathcal{G}_1$ apart from $o(N)$ exceptional colors, and apart from $o(M)$ new irregular points. Then we iterate the same procedure $K-\sqrt{K}$ times and in such a way we obtain the required TMC, a set of independent edges of distinct colors. Since K is bounded the $o(1)$ exceptional sets and probabilities cannot accumulate. On the other hand, K can be chosen arbitrarily large which assures choosing edges from $(1-\varepsilon)N$ colors for any $\varepsilon>0$, if $N>N_0(\varepsilon)$, $S>S_0(\varepsilon)$. (This means in case of $M=2N$, for example, $(1-\varepsilon)M$ chosen vertices.)

5

We will denote the set of regular colors in class t by L'^t. Let us now choose for any regular color $v \in L'^1$ an edge $(A_v, B_v) \in G(v)$ randomly with equal probability $1/S(v) = 1/|G(v)|$, and mutually independently for different values of v. Denote the set of chosen vertices by H_1, let $V_1 = V - H_1$, and consider the random variable

$$\xi_A = \begin{cases} 1 & \text{if} \quad A \notin H_1 \\ 0 & \text{if} \quad A \in H_1 \end{cases}, \qquad E(\xi_A) = P(\xi_A = 1) = P(A \notin H_1). \tag{5.1}$$

Then we have by (i″)–(iii″) for every regular vertex A,

$$\begin{aligned} E(\xi_A) = P(A \notin H_1) &= \prod_{\substack{v \in L'^1 \\ A \in V(v)}} \left(1 - \frac{1}{S(v)}\right) = e^{-\sum_{v \in L'^1,\, A \in V(v)} ((1+o(1))/S)} \\ &= e^{-(m^*(A,1)(1+o(1))/S)} = e^{-(2/cK)(1+o(1))} = e^{-2/cK} + o(1). \end{aligned} \tag{5.2}$$

If A and B $(A \neq B)$ are regular non-adjacent vertices (or connected with an edge of irregular color) then similarly to (5.2) we have

$$E(\xi_A\xi_B)=P(A\notin H_1\ \wedge\ B\notin H_1)$$

$$=\prod_{\substack{v\in L'^1\\ A,\,B\in V(v)}}\left(1-\frac{2}{S(v)}\right)\prod_{\substack{v\in L'^1\\ A\in V(v)\\ B\notin V(v)}}\left(1-\frac{1}{S(v)}\right)\prod_{\substack{v\in L'^1\\ A\notin V(v)\\ B\in V(v)}}\left(1-\frac{1}{S(v)}\right)$$

$$=e^{-(m^*(A,\,1)+m^*(B,\,1))/S(1+o(1))}$$

$$=e^{-(4/cK)(1+o(1))}=e^{-4/cK}+o(1). \tag{5.3}$$

If A and B are regular and connected with an edge of regular color $\mu\in L'^1$ then the only change is that $1-2/S(\mu)$ has to be replaced by $1-1/S(\mu)$ which does not affect the validity of $E(\xi_A\xi_B)=e^{-4/cK}+o(1)$. Further, we remark that $0\leqslant E(\xi_A)\leqslant 1$ and $0\leqslant E(\xi_A\xi_B)\leqslant 1$ hold trivially for any A and B in V.

In view of (5.1) and (5.2) the expected number of remaining vertices in the new graph $\mathcal{G}_1$ will be

$$E(|V_1|)=E(|V-H_1|)=\sum_{A\in V}E(\xi_A)=M(e^{-2/cK}+o(1)). \tag{5.4}$$

Further, by (5.1)–(5.3) and $0\leqslant E(\xi_A\xi_B)\leqslant E(\xi_A)\leqslant 1$ we have

$$D^2\left(\sum_{\xi\in V}\xi_A\right)=\sum\sum_{A,\,B\in V}E((\xi_A-E(\xi_A))(\xi_B-E(\xi_B)))$$

$$=\sum\sum_{A,\,B\in V}\{E(\xi_A\xi_B)-E(\xi_A)\,E(\xi_B)\}$$

$$=o(M^2)\cdot O(1)+(M^2-o(M^2))\,o(1)=o(M^2). \tag{5.5}$$

So with probability $(1-o(1))$ we obtain

$$|V_1|=Me^{-(2/cK)}+o(M)\leftrightarrow|H_1|=M(1-e^{-(2/cK)})+o(M). \tag{5.6}$$

Although the whole set of chosen edges (A_v,B_v) $(1\leqslant v\leqslant N/K)$ will probably not consist of completely independent edges, we can show that the overwhelming majority of (*but not almost all*) edges will be independent.

Let us consider an arbitrary regular color $v_0\in L'^1$ and consider the edge $(A,B)=(A_{v_0},B_{v_0})\in G(v_0)$. Further, let

$$\psi_A=\sharp\{C\in V;\exists v\in L'^1,(A,\,C)\in G(v),$$
$$\text{and the edge }(A,\,C)\text{ was chosen for }v\},$$

and define ψ_B similarly.

Then if A is regular we have

$$E(\psi_A) = \sum_{\substack{v \in L'^1 \\ \exists C \in V,\, (A, C) \in G(v)}} \frac{1}{S(v)} = \frac{(1+o(1))\, 2S}{cK}\, (1+o(1))\, \frac{1}{S} \leqslant \frac{1+o(1)}{K}, \quad (5.7)$$

and similar for B. Let us call a regular edge (A_v, B_v) (where $v \in L'^1$) isolated if it is independent from any other (A_μ, B_μ), where μ runs through all colors in L'^1. Then in view of (5.7),

$$\begin{aligned} P((A_v, B_v) \text{ is non-isolated}) &\leqslant P(\psi_{A_v} \geqslant 1) + P(\psi_{B_v} \geqslant 1) \\ &\leqslant E(\psi_{A_v}) + E(\psi_{B_v}) \leqslant \frac{2(1+o1))}{K}. \end{aligned}$$

Thus the expected number of isolated edges obtained in the first step will be at least

$$(1+o(1))\, \frac{N}{K} \left(1 - \frac{2(1+o(1))}{K}\right) \geqslant \frac{N}{K} \left(1 - \frac{3}{K}\right).$$

After the set H_1 of vertices is chosen we delete all edges in G which are incident to a vertex in H_1 and we will study how properties (ii″), (iii″), and (iv″) change.

Let us investigate now the set $G_1(v)$ of remaining edges of a given good color $v \notin L^1$ after we have deleted edges having a vertex in H_1. Let for any

$$(A, B) = e \in G(v) \qquad \zeta_e = \begin{cases} 1 & \text{if} \quad e \in G_1(v) \\ 0 & \text{if} \quad e \notin G_1(v) \end{cases} \qquad E(\zeta_e) = P(\zeta_e = 1). \quad (5.9)$$

Then clearly $\zeta_{(A, B)} = \zeta_A \zeta_B$ which we studied before (see (5.3)). Since almost all edges of color v are regular we obtain by (5.3)

$$\begin{aligned} E(|G_1(v)|) &= E\left(\sum_{e \in G(v)} \zeta_e\right) = o(S) \cdot O(1) + (S - o(S))(e^{-4/cK} + o(1)) \\ &= S e^{-4/cK} (1 + o(1)). \end{aligned} \quad (5.10)$$

Suppose e and f are two distinct regular edges of color v. Then by (i″) they have 4 different regular vertices A_1, A_2, A_3, A_4. With the notation $\mathscr{A} = \{A_1, ..., A_4\}$ we have similarly to (5.3)

$$\begin{aligned} E(\zeta_e \zeta_f) &= E(\xi_{A_1} \xi_{A_2} \xi_{A_3} \xi_{A_4}) = (1+o(1)) \prod_{\mu \in L'^1} \left(1 - \frac{|V(\mu) \cap \mathscr{A}|}{S(\mu)}\right) \\ &= (1+o(1))\, e^{-1/S \sum_{j=1}^4 m(A_j, 1)(1+o(1))} \\ &= (1+o(1))\, e^{-8/cK(1+o(1))} = e^{-8/cK} + o(1). \end{aligned} \quad (5.11)$$

Since almost all pairs of edges of color v are regular we have by (5.3), (5.11), and the trivial relation $0 \leqslant E(\zeta_e \zeta_f) \leqslant 1$, similar to (5.5),

$$D^2\left(\sum_{e \in G(v)} \zeta_e\right) = \sum_{e,\, f \in G(v)} \{E(\zeta_e \zeta_f) - E(\zeta_e)\, E(\zeta_f)\} = o(S^2) \cdot O(1)$$
$$+ (S^2 - o(S^2)) \cdot o(1) = o(S^2). \tag{5.12}$$

Now (5.10) and (5.12) show that we have with probability $1 - o(1)$ for any good color $v \notin L^1$ the relation

$$S_1(v) \stackrel{\text{def}}{=} |G_1(v)| = Se^{-4/cK}(1 + o(1)) \tag{5.13}$$

for the number of remaining edges of color v.

Since (5.10) and (5.12) hold uniformly for every good color $v \notin L^1$ it is easy to see that (5.13) holds with probability $1 - o(1)$ for almost all good colors $v \notin L^1$ and so for almost all colors $v \notin L^1$.

In order to see this consider any fixed small $\delta > 0$ and let for a good $v \notin L^1$

$$\eta_v = \begin{cases} 1 & \text{if} \quad |S_1(v) - Se^{-4/cK}| > \delta S \\ 0 & \text{otherwise} \end{cases}. \tag{5.14}$$

Then by (5.10) and (5.12) we have from Chebyshev's inequality

$$E(\eta_v) = P(\eta_v = 1) = o(1). \tag{5.15}$$

Further, we obtain for every pair of good colors $v, \mu \notin L^1$ trivially by (5.15)

$$E(\eta_v \eta_\mu) = P(\eta_v = \eta_\mu = 1) = o(1), \tag{5.16}$$

and so we have in view of (5.15) and (5.16)

$$E\left(\sum_{v \notin L^1} \eta_v\right) = o(N), \qquad D^2\left(\sum_{v \notin L^1} \eta_v\right) = o(N^2), \tag{5.17}$$

which implies

$$p\left(\sum_{v \notin L^1} \eta_v > \delta N\right) < \delta \qquad \text{if} \quad N > N(\delta),\, S > S(\delta). \tag{5.18}$$

In the following we will study how many edges belonging to a given color class L^t $(t \neq 1)$ and incident at a given regular point $A \in V_1$ will remain after deleting those edges which have a vertex in H_1.

Let us define for any $e=(A, B)\in G^t (t\neq 1)$ similarly to (5.1)

$$\zeta_e^* = \xi_B = \begin{cases} 1 & \text{if} \quad B \notin H_1 \\ O & \text{if} \quad B \in H_1 \end{cases}, \qquad E(\xi_B) = P(\xi_B = 1). \tag{5.19}$$

Then we have for any good point A by (ii″) and (5.2)

$$\begin{aligned} E(m_1(A, t)) &= E\left(\sum_{(A, B)\in G^t} \xi_B\right) \\ &= o(S)\cdot O(1) + \left((1+o(1))\frac{2S}{cK}\right)\left(e^{-2/cK}+o(1)\right) \\ &= \frac{2S}{cK}e^{-2/cK} + o(S), \end{aligned} \tag{5.20}$$

where m_1 is defined for $\mathcal{G}_1$ similarly to m for G.

Further, for any fixed good point A we have by (5.3), analogously to (5.5)

$$\begin{aligned} D^2\left(\sum_{(A, b)\in G^t} \xi_B\right) &= \sum \sum_{A,\, B\in G^t,\, (A, B')\in G^t} \{E(\xi_B \xi_{B'}) - E(\xi_B)\, E(\xi_{B'})\} \\ &= o(S^2)\, O(1) + O(S^2)\, o(1) = o(S^2). \end{aligned} \tag{5.21}$$

Therefore we have for the number of remaining edges of a given color class L^t $(t\neq 1)$ incident at a given good point $A\in V_1$ with probability $1-o(1)$, the relation

$$m_1(A, t) = \frac{2S}{cK}e^{-2/cK} + o(S). \tag{5.22}$$

Similarly to the argument (5.14)–(5.18) we obtain that for any $\delta>0$ with the notation

$$\omega_{A,\, t} = \begin{cases} 1 & \text{if} \quad |m_1(A, t) - (2S/cK)\, e^{-2/cK}| > \delta S \\ 0 & \text{otherwise} \end{cases}, \tag{5.23}$$

we have

$$P\left(\sum_{t=2}^{k} \sum_{A \text{ good}} \omega_{A,\, t} > \delta M\right) < \delta \qquad \text{if} \quad N > N_0(\delta),\ S > S_0(\delta). \tag{5.24}$$

This means that with probability $1-o(1)$ almost all good points in V_1 and therefore almost all points $A \in V_1$ satisfy the relation (5.22) for every value $t=2, 3, ..., K$. Finally, we have trivially

$$m_1(A) \leqslant m(A) \leqslant \frac{2FS}{c}. \tag{5.25}$$

6

After the random choice of H_1 we delete in V_1 all colors in the first color class. The number of remaining colors will be in each remaining color-class $t=2, ..., K$ unchanged,

$$\frac{N}{K}+o(N), \tag{6.1}$$

where the number of classes will be $K_1=K-1$.

According to (5.13) we have in almost all not deleted colors $S_1(1+o(1))$ edges where

$$S_1=Se^{-4/cK}. \tag{6.2}$$

Taking into account (5.6) and (6.1) we have a graph with $M_1(1+o(1))$ vertices and $N_1(1+o(1))$ colors where

$$M_1=Me^{-2/cK}, \qquad N_1=N\frac{K-1}{K}=N\frac{K_1}{K}, \tag{6.3}$$

and so by $c \geqslant 2$ we have

$$c_1=\frac{M_1}{N_1}=c\frac{c^{-2/cK}}{1-1/K} \geqslant c\frac{e^{-1/K}}{1-1/K} > c. \tag{6.4}$$

Further, we have from (6.4)

$$c_1 < c\frac{1}{1-1/K}=c \cdot \frac{K}{K_1} \qquad \text{and} \qquad c_1K_1=cKe^{-2/cK}. \tag{6.5}$$

In view of (5.22), (5.25), and (6.5) we have for almost all $A \in V_1$ and all $t \neq 1$

$$m_1(A, t)(1+o(1))=\frac{2Se^{-4/cK}}{cKe^{-2/cK}}=\frac{2S_1}{c_1K_1}. \tag{6.6}$$

On the other hand, we have for any $A \in V_1$, $t \neq 1$ by (5.25), (6.2), (6.4), and (6.5)

$$m(A) \leqslant \frac{2F_1 S_1}{c_1} \tag{6.7}$$

where

$$\frac{F_1}{F} = \frac{S}{S_1} \cdot \frac{c_1}{c} = e^{2/cK} \frac{1}{1-1/K} < e^{1/K} \cdot \frac{K}{K-1} < \left(\frac{K}{K-1}\right)^2.$$

In such a way we see that with probability $1-o(1)$ the new graph $\mathcal{G}_1$ will satisfy the conditions (i″)–(iv″) with the new values of the parameters $M_1, N_1, c_1, K_1, S_1, F_1$ given by (6.2)–(6.7). So we can iterate this procedure a bounded number of time. Making $K-\sqrt{K}$ steps we will have during the whole procedure by (6.5) and (6.7) always

$$c_\nu < c\,\frac{K}{K_1} \cdot \frac{K_1}{K_2} \cdots \frac{K_{\nu-1}}{K_\nu} = c\,\frac{K}{K-\nu} \leqslant c\sqrt{K}, \qquad F_\nu < F\left(\frac{K}{K-\nu}\right)^2 \leqslant FK, \tag{6.8}$$

so both c_ν and F_ν (and naturally $K_\nu = K-\nu$) will remain bounded since c, F, and K were bounded. Further, by (6.2), (6.3) and (6.5) we have for all $\nu \leqslant K-\sqrt{K}$

$$\begin{aligned} N_\nu &= N\,\frac{K_1}{K} \cdot \frac{K_2}{K_1} \cdots \frac{K_\nu}{K_{\nu-1}} = N\,\frac{K-\nu}{K} \geqslant \frac{N}{\sqrt{K}}, \\ S_\nu &= S\left(\frac{K_1 c_1}{Kc}\right)^2 \cdot \left(\frac{K_2 c_2}{K_1 c_1}\right)^2 \cdots \left(\frac{K_\nu c_\nu}{K_{\nu-1} c_{\nu-1}}\right)^2 = S\left(\frac{K_\nu c_\nu}{Kc}\right)^2 > S\left(\frac{K_\nu}{K}\right)^2 \geqslant \frac{S}{K}. \end{aligned} \tag{6.9}$$

This means that $M_\nu > N_\nu > N(K)$, $S_\nu > S(K)$ can be assured during the whole procedure by a suitable choice of $N_0(K)$ and $S_0(K)$. Further, the meanings of $o(N_\nu)$ and $o(N)$, resp. $o(S_\nu)$ and $o(S)$, are the same.

Finally according to (5.8) the expected number of independent edges with distinct colors chosen from the color-class $\nu \leqslant K-\sqrt{K}$ is at least

$$\begin{aligned} (1+o(1))\,\frac{N_{\nu-1}}{K_{\nu-1}}\left(1-\frac{3}{K_{\nu-1}}\right) &= (1+o(1))\,\frac{N}{K}\left(1-\frac{3}{K_{\nu-1}}\right) \\ &\geqslant (1+o(1))\,\frac{N}{K}\left(1-\frac{3}{\sqrt{K}}\right). \end{aligned} \tag{6.10}$$

Thus after $K-\sqrt{K}$ steps we expect to obtain at least

$$(1+o(1))(K-\sqrt{K}) \cdot \frac{N}{K}\left(1-\frac{3}{\sqrt{K}}\right) > N\left(1-\frac{4}{\sqrt{K}}\right) \tag{6.11}$$

independent edges of distinct colors. So we have the required TMC partial matching.

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